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Free Vibrations and Sensitivity Analysis of a Defective Two Degree-of-Freedom System

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Free oscillations of a two degree-of-freedom system with nonproportional damping are analyzed. By a suitable choice of parameters, a family of defective systems having a noncomplete set of eigenvectors is selected. Free motions of underdamped and overdamped defective systems are studied in the four-dimensional state space, and their main characteristics are discussed. In particular, the rate at which the trajectories are attracted by the eigenvectors is determined. Small perturbations of order ε of the parameters are then considered, and asymptotic expressions for the modified system eigensolutions are obtained. These allow qualitative discussion of the effects of modifications on the mechanical behavior of nearly defective systems. Marked sensitivities of order $\varepsilon^{\frac{1}{2}}$ or $\varepsilon^{\frac{1}{4}}$ are found. These depend strongly on the damping magnitude. An extensive numerical analysis is performed.

I. Introduction

IN structural dynamics it is by no means rare to encounter systems having multiple eigenvalues. This does not pose any substantial difficulties when the system is conservative, but it involves specific problems when the system is nonconservative. Very often indeed, the geometric multiplicity of the eigenvalue is less than the algebraic multiplicity, and so the system has an incomplete set of eigenvectors, insufficient to form a base for the state space. Systems of this type are called defective.

The free evolution of defective systems is well known to persons working in the automatic control or system theory field; moreover, basic notions can be found in any good book on linear algebra. However, in the author's opinion, the problem is not sufficiently known to the structural analyst. Indeed, it is common practice to assume damping of proportional type, so that the eigenvectors coincide with those of the corresponding undamped system (always forming a complete set), and defective systems cannot occur.

Simple examples of defective mechanical systems can certainly be given. First, a one degree-of-freedom (DOF) oscillator with critical damping $\zeta = 1$ is a defective system, since a unique eigenvector $\{\dot{x}, x\} = \{-\omega_0, 1\}$ is associated with the double eivenvalue $\lambda = -\omega_0$, ω_0 being the natural undamped frequency of the oscillator; motion then develops with a mixed algebraic-exponential time law. As a second example let us consider a structure in the critical flutter condition, in which two couples of imaginary eigenvalues $\lambda = \pm i\omega$ coincide. Usually a unique couple of complex conjugate eigenvectors exists, and so the solution is still algebraic exponential. A third, more general, defective system can be constructed starting from a conservative system having many close frequencies, e.g., a band spectrum system. By introducing in suitable points several light dampers and/or small follower forces, it is possible, in principle, to render all of the eigenvalues in a band equal.

A conclusion can be drawn from previous examples; namely, if it is fairly easy to build a defective system, it is equally easy to modify it by varying the project parameters to make the system nondefective. Therefore it is very important to know the modal sensitivities of such a system to evaluate the eigensolutions of the modified system. However, standard methods (e.g., treated in the book by Brandon[1]) fail in the defective case, and so it is necessary to resort to particular techniques. The problem has been discussed by the author in Ref. 2, where a perturbation algorithm for evaluating the eigenderivatives of defective systems is proposed.

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The main aims of this paper are the following: 1) to show that mechanical defective systems can really exist if nonconservative forces are take into account; 2) to show that these systems are represented in the parameters space not by isolated points but rather by curves or surfaces, i.e., they form one-parameter or multiparameter families; 3) to discuss the main characteristics of these pathological systems in connection with the strength of the nonconservative forces; 4) to analyze the mechanical properties of nearly defective systems by using sensitivity analysis. To this end use is made of a simple model. The free vibrations of a two DOF oscillator with nonproportional damping are studied. By appropriate selection of parameters the system is made defective or nearly defective. Phase space representations and eigenvalues loci are employed to discuss its dynamical properties.

II. Equations of Motion

Let us consider the nonconservative two DOF system illustrated in Fig. 1. It consists of two simple oscillators of stiffness k_j and mass m_j $(j = 1, 2)$, coupled by a linear damper of constant c. By denoting by q_j the displacement of the jth mass and defining the following dimensionless parameters,

$$\mu = m_2/m_1 > 1, \qquad \kappa = k_2/k_1, \qquad \zeta = \frac{c}{2\sqrt{k_1 m_1}} \tag{1}$$

the equations of motion read

$$\begin{aligned} \ddot{q}_1 + 2\zeta(\dot{q}_1 - \dot{q}_2) + q_1 &= 0 \\ \mu\ddot{q}_2 + 2\zeta(\dot{q}_2 - \dot{q}_1) + \kappa q_2 &= 0 \end{aligned} \tag{2}$$

where the dot denotes differentiation with respect to the dimensionless time $t = (k_1/m_1)^{\frac{1}{2}}\,\tilde{t}$. Equations (2) can be put in matrix form

$$\dot{x} = Ax \tag{3}$$

in which

$$x = \{\dot{q}_1, \dot{q}_2, q_1, q_2\}^T \tag{4}$$

is the 4×1 state vector and

$$A = \left[\begin{array}{cc|cc} -2\zeta & 2\zeta & -1 & 0 \\ 2\zeta/\mu & -2\zeta/\mu & 0 & -\kappa/\mu \\ \hline 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{array}\right] \tag{5}$$

is the 4×4 dynamic matrix of the system.

The general solution of the equations of motion (3) can be expressed in terms of the eigensolutions of the matrix A. Here interest is focused on the case in which the dynamic matrix 1) does not have

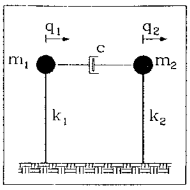


Fig. 1 Model.

a complete set of eigenvectors or 2) does, in fact, have a complete set of eigenvectors, but some of them are nearly parallel. Systems of type 1 will be referred to as defective, systems of type 2 as nearly defective.

III. Defective Systems

The dynamic matrix (5) is a function of the three parameters, Eqs. (1), i.e., $A = A(\mu, \kappa, \zeta)$. For particular values $\mu = \mu_0$, $\kappa = \kappa_0$, $\zeta = \zeta_0$, the matrix $A_0 = A(\mu_0, \kappa_0, \zeta_0)$ has eigenvalues λ_0 of algebraic multiplicity $m > 1$. If the associated geometric multiplicity n is less than m, A_0 is defective.

It is easy to check that if

$$\mu_0 = (1+\zeta_0)/(1-\zeta_0), \qquad \kappa_0 = 1/\mu_0 \tag{6}$$

A_0 admits two double eigenvalues

$$\lambda_{0k} = \frac{-\zeta_0 \pm i\sqrt{1-2\zeta_0^2}}{1+\zeta_0} \qquad (k = 1, 2) \tag{7}$$

a unique eigenvector

$$u_{1k} = \left[\frac{2\zeta_0\lambda_{0k}^2}{1+2\zeta_0\lambda_{0k}+\lambda_{0k}^2}, \lambda_{0k}, \frac{2\zeta_0\lambda_{0k}}{1+2\zeta_0\lambda_{0k}+\lambda_{0k}^2}, 1\right]^T$$
$$(k = 1, 2) \tag{8}$$

being associated with each of them. Therefore matrix A_0 is defective.

By assuming, for example, ζ_0 as an independent variable, Eqs. (6) select a one-parameter family of defective systems. It should be noted that when $\zeta_0 \to 0$, then $\mu_0 \to 1$, $\kappa_0 \to 1$, and $\lambda_0 \to \pm i$; the system degenerates into two uncoupled identical oscillators, i.e., a nondefective system. On the other hand, when $\zeta_0 \to 1$, then $\mu_0 \to \infty$ and $\kappa_0 \to 0$, i.e., the oscillator $j = 2$ degenerates into a free mass of infinite magnitude.

From Eqs. (7) it follows that 1) when $0 < \zeta_0 < \sqrt{2}/2$, there are two complex conjugate double eigenvalues (underdamped system); 2) when $\sqrt{2}/2 < \zeta_0 < 1$, there are two real double eigenvalues (overdamped system); and 3) when $\zeta_0 = \zeta_{0_{cr}} = \sqrt{2}/2$, there is a unique real quadruple eigenvalue (critically damped system).

In cases 1 and 2, $m = 2$ and $n = 1$ for each λ_{0k} $(k = 1, 2)$. The two eigenvectors u_{1k} do not form a base for the four-dimensional space; however, it is possible to complete this base by determining two generalized eigenvectors of order (or index) 2, u_{2k}, that satisfy the following equations (e.g., see Ref. 3)

$$(A_0 - \lambda_{0k} I)u_{2k} = u_{1k} \qquad (k = 1, 2) \tag{9}$$

Eigenvectors u_{1k} and u_{2k} are said to constitute a chain of generalized eigenvectors of length 2. Matrix $U = [u_{11}, u_{21}, u_{12}, u_{22}]$ transforms A_0 by similarity into the Jordan canonical form, made of two 2×2 blocks on the diagonal. By solving the equations of motion in the new base and then coming back to the original base, the following general solution is determined:

$$x(t) = \sum_{k=1}^{2} e^{\lambda_{0k}t}[c_{1k}u_{1k} + c_{2k}(u_{2k} + tu_{1k})] \tag{10}$$

Constants c_{jk} $(j, k = 1, 2)$ are equal to the components of the vector $x_0 = x(0)$ in the base $\{u_{ik}\}$. By denoting with $\{v_{ik}\}$ the reciprocal base, formed by the generalized left eigenvectors, since $u_{jl}^H v_{ik} = \delta_{ij}\,\delta_{kl}$, it ensues that $c_{jk} = v_{jk}^H x_0$, where $()^H$ denotes the transpose conjugate and δ_{ij} is the Kronecker symbol.

In case 2 all quantities in Eq. (10) are real; in case 1 they are complex, conjugate in twos: $u_{j2} = \bar{u}_{j1}$, $c_{2j} = \bar{c}_{j1}$. By omitting the second index $k = 1$ and by posing $\lambda_0 = \alpha_0 + i\omega_0$, $c_j = a_j \exp(i\varphi_j)$, and $u_j = y_j + iz_j$, Eq. (10) reads

$$x(t) = 2e^{\alpha_0 t}\{a_1[y_1\cos(\omega_0 t + \phi_1) - z_1\sin(\omega_0 t + \phi_1)]$$
$$+ a_2[(y_2 + ty_1)\cos(\omega_0 t + \phi_2) - (z_2 + tz_1)\sin(\omega_0 t + \phi_2)]\} \tag{10'}$$

In case 3 it ensues that $m = 4$ and $n = 1$, i.e., a unique eigenvector u_1 (by omitting the second index $k = 1$) is associated with the quadruple eigenvalue λ_0. In this case three generalized eigenvectors u_j $(j = 2, 3, 4)$ have to be determined by the recurrent relations

$$(A_0 - \lambda_0 I)u_j = u_{j-1} \qquad (j = 2, 3, 4) \tag{11}$$

By proceeding as in the previous case, the following solution is obtained:

$$x(t) = e^{\lambda_0 t}\left[c_1u_1 + c_2(u_2 + tu_1) + c_3\left(u_3 + tu_2 + \frac{t^2}{2}u_1\right)\right.$$
$$\left. + c_4\left(u_4 + tu_3 + \frac{t^2}{2}u_2 + \frac{t^3}{6}u_1\right)\right] \tag{12}$$

where $c_i = v_i^H x_0$ and all quantities are real.

It should be noted that, in all cases, generalized eigenvectors are not univocally determined by Eqs. (9) or (11), since the operator $A_0 - \lambda_0 I$ is singular. However, arbitrary quantities can be avoided by introducing a suitable chosen normalization condition, e.g., by requiring all of the elements of a chain to be orthogonal to the first element, i.e., $u_{jk}^H u_{1k} = 0$ when $j \geq 2$. This condition has been utilized before.

In the following the dynamics of the defective system are analyzed for the cases previously described. Greater attention is paid to the undercritical case, since this is the most important from a technical point of view.

A. Undercritical Damping

The analytical expressions of the generalized eigenvectors are rather involved, and so it is better to evaluate them numerically. However, when $\zeta_0 \ll 1$, it is possible to obtain the following simple asymptotic solution:

$$u_1 = \begin{bmatrix} -1 \\ i \\ i \\ 1 \end{bmatrix} + \zeta_0 \begin{bmatrix} 1-i \\ -1-i \\ 0 \\ 0 \end{bmatrix} + \mathcal{O}(\zeta_0^2) \tag{13a}$$

$$u_2 = \frac{1}{2\zeta_0}\begin{bmatrix} i \\ -1 \\ 1 \\ i \end{bmatrix} + \frac{1}{2}\begin{bmatrix} -1+i \\ 1 \\ -i \\ 1+2i \end{bmatrix} + \mathcal{O}(\zeta_0) \tag{13b}$$

By assembling matrix $U = [u_1, u_2, \bar{u}_1, \bar{u}_2]$ and inverting it, matrix $V^H = [v_1^H, v_2^H, \bar{v}_1^H\, \bar{v}_2^H] \equiv U^{-1}$ is obtained, where

$$v_1 = \frac{1}{4}\begin{bmatrix} -1 \\ i \\ i \\ 1 \end{bmatrix} + \frac{\zeta_0}{4}\begin{bmatrix} -1 \\ i \\ -1 \\ i \end{bmatrix} + \mathcal{O}(\zeta_0^2)$$
$$v_2 = \frac{\zeta_0}{2}\begin{bmatrix} i \\ -1 \\ 1 \\ i \end{bmatrix} + \frac{\zeta_0^2}{2}\begin{bmatrix} 1-i \\ -1+i \\ 0 \\ -2i \end{bmatrix} + \mathcal{O}(\zeta_0^3) \tag{14}$$

are the left generalized eigenvectors; v_2 is the proper left eigenvector.

Expansions (13) show that u_1 and u_2 coincide with the proper eigenvectors of the nondefective degenerate system $\zeta_0 = 0$, if only the leading terms of the expansions are considered. In addition it is seen that $\|u_1\| = (\bar{u}_1, u_1)^{\frac{1}{2}} = \mathcal{O}(1)$ and $\|u_2\| = \mathcal{O}(\zeta_0^{-1})$, and thus $\|u_2\| \to \infty$ when $\zeta_0 \to 0$. However, if Eqs. (10′) are considered,

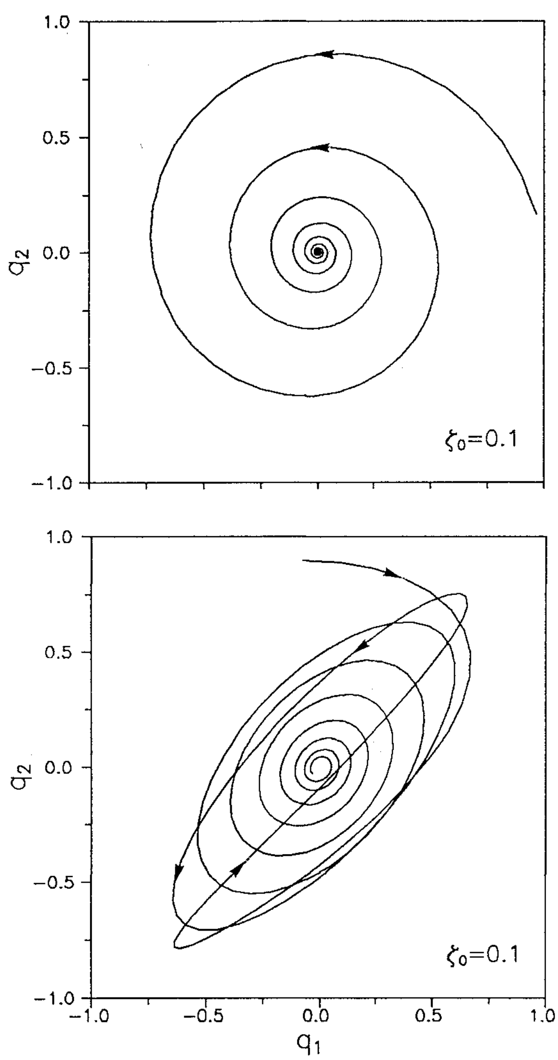


Fig. 2 Underdamped system: a) amplitude-decaying periodic motion and b) nonperiodic motions.

constants $a_j = \|v_j^H x_0\|$ are found to be $a_1 = \mathcal{O}(1)$ and $a_2 = \mathcal{O}(\zeta_0)$ if $\|x_0\| = \mathcal{O}(1)$; thus the two terms in the square brackets are of the same order, and the solution tends correctly to the solution of the nondefective system when $\zeta_0 \to 0$ and $t < \mathcal{O}(\zeta_0^{-1})$.

Equation (10′) shows that a unique mode exists, depending on four arbitrary constants. The motion is a sum of two component motions: the first one, of amplitude a_1, is an amplitude decaying ($\alpha_0 < 0$) periodic motion, the trajectory of which is a spiral lying on the plane spanned by y_1 and z_1; the other one, of amplitude a_2, is an aperiodic motion whose trajectory runs through the whole four-dimensional state space. However, this trajectory tends to the plane (y_1, z_1) for $t \to \infty$, although the tendency is weak, of polynomial type, compared with the exponential decay of the motion. Because of the order of magnitude of u_1 and u_2 the distance of the trajectory from the plane comes to be of order ζ_0 when $t = \mathcal{O}(\zeta_0^{-2})$.

The projections on the plane (q_1, q_2) of two trajectories of the previously described types are shown in Fig. 2. In Fig. 2a it is apparent that, according to Eq. (13a), q_1 and q_2 are shifted $\pi/2$ rad out of phase. In Fig. 2b it is noted that the trajectory tends to assume a shape similar to the previous one, after having exhausted a transient phase.

B. Critical and Overcritical Damping

In the overcritical case the law of motion is given by Eq. (10). There are two modes ($k = 1, 2$), each depending on two arbitrary constants. Each mode is a sum of a monodimensional motion exponentially decaying and a bidimensional motion decaying with an algebraic-exponential law.

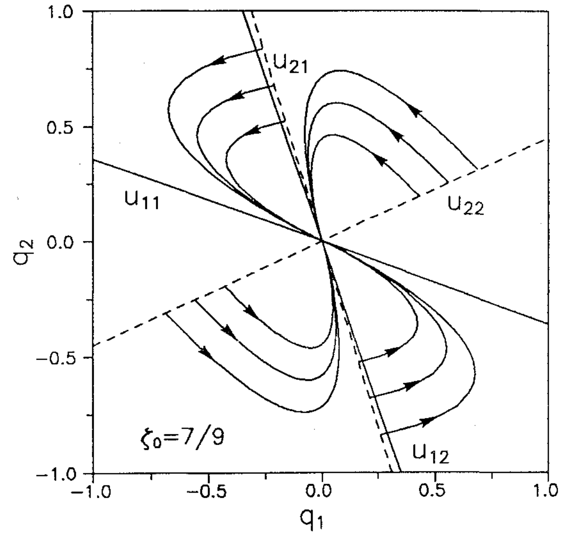


Fig. 3 Overdamped system motions.

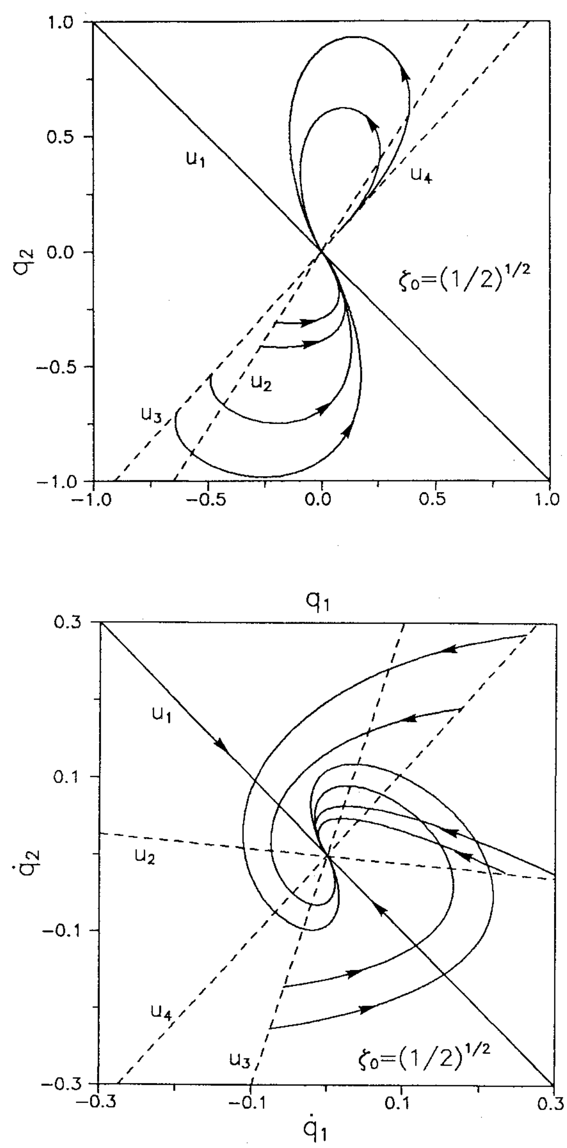


Fig. 4 Critically damped system motions: a) configurations plane and b) velocities plane.

In Fig. 3 the proper eigenvectors u_{11} and u_{12} (solid lines) and the order-two generalized eigenvectors u_{21} and u_{22} (dashed lines) of an overdamped system are represented on the plane (q_1, q_2). Some trajectories arising from points belonging to the dashed lines are also plotted. All of the trajectories are attracted by the proper eigenvectors.

In the critical case the motion is described by Eq. (12), and a unique mode dependent on four arbitrary constants exists. Motion is a sum of four component motions, developing in subspaces of increasing dimensions. The four generalized eigenvectors are represented in Fig. 4a on the (q_1, q_2) plane and in Fig. 4b on the $(\dot{q}_1, \dot{q}_2)$ plane; in the configuration plane, u_3 and u_4 have the same components. All of the motions arising from points along the generalized eigenvector directions are slowly attracted by the unique proper eigenvector.

IV. Nearly Defective Systems

Let us now perturb the values of the parameters that make the system defective, i.e., let

$$\mu = \mu_0 + \varepsilon\mu_1, \qquad \kappa = \kappa_0 + \varepsilon\kappa_1, \qquad \zeta = \zeta_0 + \varepsilon\zeta_1 \quad (15)$$

where $\varepsilon \ll 1$ is a perturbation parameter. It will be supposed here that matrix $A[\mu(\varepsilon), \kappa(\varepsilon), \zeta(\varepsilon)] = A(\varepsilon)$ is nondefective for any $\varepsilon \neq 0$ in the interval of interest.

The equations of motion (3) admit particular solutions $x(t) = w_k \exp(\lambda_k t)$, where w_k and λ_k are eigensolutions of A:

$$(A - \lambda_k I)w_k = 0 \quad (16)$$

Since, by hypothesis, the set w_k is complete, the general solution of Eqs. (3) reads

$$x(t) = \sum_{k=1}^{n_r} a_k y_k e^{\alpha_k t} + 2\sum_{k=1}^{n_c} a_k e^{\alpha_k t}$$
$$\times [y_k \cos(\omega_k t + \varphi_k) - z_k \sin(\omega_k t + \varphi_k)] \quad (17)$$

where $\lambda_k = \alpha_k + i\omega_k$, $w_k = y_k + iz_k$, n_r is the number of real eigenvalues, and $2n_c$ is the number of complex conjugate eigenvalues, with $n_r + 2n_c = 4$. The arbitrary constants are determined by the initial conditions, as $a_k \exp(i\varphi_k) = v_k^H x_0$, where v_k is the kth (proper) left eigenvalue of A.

The eigensolutions of matrix $A(\varepsilon)$ are themselves a function of ε. Obviously they could be easily determined numerically for various ε, as the dimensions of the problem in question are very small. However, in more general cases, it is much more convenient to evaluate their derivatives at $\varepsilon = 0$ (modal sensitivities) and then extrapolate their values for $\varepsilon \neq 0$ by means of Taylor's expansion. If this way is chosen, as a first step matrix $A(\varepsilon)$ must be expanded around $\varepsilon = 0$:

$$A = A_0 + \varepsilon A_1 + \varepsilon^2 A_2 + \cdots \quad (18)$$

then $w(\varepsilon)$ and $\lambda(\varepsilon)$ must themselves be expanded around an eigensolution of A_0. However, since A_0 is defective, standard methods employing expansions of integer powers of ε fail and specific techniques must be adopted. The general problem has been solved by the author in Ref. 2 by using series expansions of noninteger powers of the perturbation parameter, according to ideas contained in Refs. 4 and 5. The algorithm proposed allows the consideration of general modifications, provided they depend on a unique parameter, and therefore generalizes the (exact) method of Pomazal and Snyder[6] that studies only the local modification problem. Here the simplest case of a defective system occurs, i.e., the case in which only a chain of generalized eigenvectors is associated with a multiple eigenvalue. It is sufficient, therefore, to expand the eigensolutions in series of $\varepsilon^{1/m}$, where m is the algebraic multiplicity of $\lambda_0 = \lambda(0)$. Both not critically damped and critically damped defective systems are studied next.

A. Not Critically Damped Systems

In this case $m = 2$; the following expansion is then performed:

$$\lambda_k = \lambda_{0k}\varepsilon^{\frac{1}{2}}\lambda_{1k} + \varepsilon\lambda_{2k} + \mathcal{O}(\varepsilon^{\frac{3}{2}}) \quad (19a)$$

$$w_k = w_{0k} + \varepsilon^{\frac{1}{2}} w_{1k}\varepsilon w_{2k} + \mathcal{O}(\varepsilon^{\frac{3}{2}}) \quad (19b)$$

where $(\lambda_{0\kappa}, w_{0\kappa} \equiv u_{1\kappa})$, is the kth eigensolution ($k = 1, 2$) of the unperturbed system. By using Eqs. (16), (18), and (19), the following perturbation equations are derived:

$$\varepsilon\colon (A_0 - \lambda_0 I)w_0 = 0 \quad (20a)$$

$$\varepsilon^{\frac{1}{2}}\colon (A_0 - \lambda_0 I)w_1 = \lambda_1 w_0 \quad (20b)$$

$$\varepsilon\colon (A_0 - \lambda_0 I)w_2 = \lambda_1 w_1 + \lambda_2 w_0 - A_1 w_0 \quad (20c)$$

$$\varepsilon^{\frac{3}{2}}\colon (A_0 - \lambda_0 I)w_3 = \lambda_1 w_2 + \lambda_2 w_1 + \lambda_3 w_0 - A_1 w_1 \quad (20d)$$

where, for simplicity, index k has been omitted. Since each of Eqs. (20) admits ∞^1 solutions, the normalization conditions $u_1^H w_j = 0$ ($j = 1, 2, \ldots$) are added to them to determine the arbitrary constants.

Equation (20a) is satisfied. By using Eq. (9), $w_1 = \lambda_1 u_2$ is obtained from Eq. (20b). Equation (20c) then reads

$$(A_0 - \lambda_0 I)w_2 = \lambda_1^2 u_2 - A_1 u_1 + \lambda_2 u_1 \quad (21)$$

For the solvability of this equation, the right hand member must be made orthogonal to the (proper) left eigenvector v_2. Bearing in mind that $v_2^H u_2 = 1$ and $v_2^H u_1 = 0$, then

$$\lambda_1 = \left(v_2^H A_1 u_1\right)^{\frac{1}{2}} \quad (22)$$

from which two complex roots λ_1 are obtained. Then, by solving Eq. (21).

$$w_2 = \hat{w}_2 + \lambda_2 u_2 \quad (23)$$

is obtained, where $\hat{w}_2$ is the (unique) solution of the problem

$$(A_0 - \lambda_0 I)\hat{w}_2 = \lambda_1^2 u_2 - A_1 u_1$$
$$u_1^H \hat{w}_2 = 0 \quad (24)$$

By substituting the results in Eq. (20d), from the solvability condition

$$\lambda_2 = \left(v_2^H A_1 u_2 - v_2^H \hat{w}_2\right)/2 \quad (25)$$

is derived. To summarize: any eigenvalue λ_0 of the unperturbed defective system generates two eigenvalues of the perturbed nondefective system. The eigenvectors lie on the complex plane near a circumference of center λ_0 and radius of order $\varepsilon^{\frac{1}{2}}$. The eigenvalues are nearly parallel and differ from the generator eigenvector u_1 by small quantities of order $\varepsilon^{\frac{1}{2}}$ along the u_2 direction. It should be noted that these differences do not depend on the perturbation shape matrix A_1, which in fact appears only at the ε order. The system exhibits high modal sensitivity of order $\varepsilon^{\frac{1}{2}}$.

The two-term solution (second order or $\varepsilon^{\frac{1}{2}}$) determined in this way usually furnishes a good quantitative approximation of the exact solution of Eqs. (16) (see Ref. 2). It requires knowledge of the generalized eigenvectors of A_0 that are only known numerically. However, use can be made of the asymptotic expressions (13) and (14) to obtain qualitative information, although strictly valid only for $\zeta_0 \to 0$. By proceeding in this manner, the following points can be made:

1) Since $\|v_2\| = \mathcal{O}(\zeta_0)$, $\|u_1\| = \mathcal{O}(1)$, and $\|A_1\| = \mathcal{O}(1)$, from Eqs. (22) then $\lambda_1 = \mathcal{O}(\zeta_0^{\frac{1}{2}})$. Thus, the sensitivity of the eigenvalues depends on the magnitude of the nonconservative component of the system. Weakly nonconservative defective systems ($\zeta_0 \ll 1$) exhibit marked sensitivity only in a small neighborhood of $\varepsilon = 0$, and so they can be classified as moderately sensitive

systems. On the contrary, moderately or strongly nonconservative defective systems manifest high sensitivity in a larger neighborhood of $\varepsilon = 0$; therefore they can be classified as strongly sensitive to perturbations.

2) Remembering that $\lambda_1 = \mathcal{O}(\zeta_0^{\frac{1}{2}})$ and $\|u_2\| = \mathcal{O}(\zeta_0^{-1})$, the first-order component of the eigenvector w is of $(\varepsilon/\zeta_0)^{\frac{1}{2}}$ order. Therefore, the eigenvectors of weakly nonconservative defective systems are strongly sensitive to modifications, whereas the eigenvectors of strongly nonconservative defective systems are moderately sensitive. Thus eigenvector behave in opposite ways to eigenvalues.

B. Critically Damped System

In this case $m = 4$; the expansion of the eigensolution is carried out as follows:

$$\lambda = \lambda_0 + \varepsilon^{\frac{1}{4}}\lambda_1 + \varepsilon^{\frac{1}{2}}\lambda_2 + \mathcal{O}\left(\varepsilon^{\frac{3}{4}}\right) \tag{26a}$$

$$w = w_0 + \varepsilon^{\frac{1}{4}}w_1 + \varepsilon^{\frac{1}{2}}w_2 + \mathcal{O}\left(\varepsilon^{\frac{3}{4}}\right) \tag{26b}$$

By applying the procedure previously illustrated, the following results are obtained:

$$\begin{aligned} &w_0 = u_1, \qquad w_1 = \lambda_1 u_2, \qquad w_2 = \lambda_1^2 u_3 + \lambda_2 u_2 \\ &\lambda_1 = \left(v_4^H A_1 u_1\right)^{\frac{1}{4}}, \qquad \lambda_2 = \left(v_4^H A_1 u_2 - v_4^H \hat{w}_4\right)/4\lambda_1^2 \end{aligned} \tag{27}$$

where λ_1 and λ_2 are determined by the solvability conditions of the ε and $\varepsilon^{\frac{5}{4}}$ order perturbation equations. In Eq. (27) v_4 is the (unique) proper left eigenvector, and $\hat{w}_4$ is the (unique) solution of the problem

$$\begin{aligned} (A_0 - \lambda_0 I)\hat{w}_4 &= \lambda_1^4 u_4 - A_1 u_1 \\ u_1^H \hat{w}_4 &= 0 \end{aligned} \tag{28}$$

By using Eqs. (26–28), four eigensolutions of the perturbed problem can be found, one for each complex root λ_1. All of the eigensolutions are generated by the unique eigensolution of the unperturbed problem. Modal sensitivity is of $\varepsilon^{\frac{1}{4}}$ order.

V. Numerical Results

The eigenvalue problem (16) is numerically solved for different values of the parameters. Some numerical and perturbative solutions are compared.

A. Damping Modification

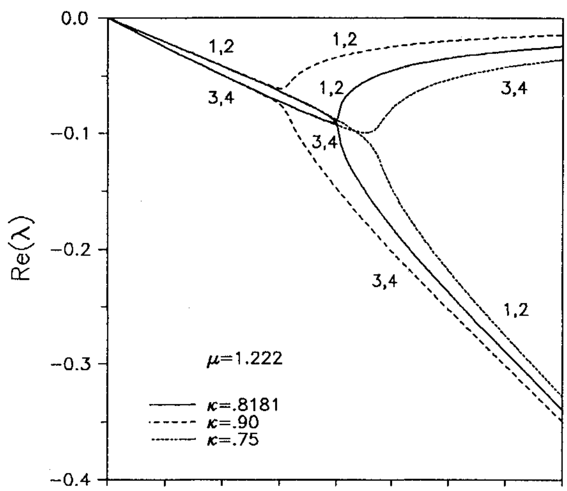
Modifications of the damping coefficient only are considered first. The analysis, besides having an intrinsic value, throws light on the mechanism that makes the nonconservative system defective.

Starting from the defective system with parameters $\zeta_0 = 1/10$, $\mu_0 = 11/9$, and $\kappa_0 = 9/11$, and keeping $\mu = \mu_0$ and $\kappa = \kappa_0$ steady, the damping ζ is varied in the interval [0, 0.2]; the locus $\lambda(\zeta)$ obtained is illustrated in Fig. 5 by solid lines. When $\zeta = 0$, there are four pure imaginary roots, conjugate in twos, corresponding to the two natural frequencies of the uncoupled degenerate system. When ζ increases, the roots become complex, conjugate in twos, until two double roots appear for $\zeta = \zeta_0 = 0.1$. In this condition a bifurcation point occurs in the locus and, correspondingly, the system becomes defective. When $\zeta > \zeta_0$, the roots again become separated. A moderate sensitivity of the eigenvalues should be noted around the bifurcation point; this is due to the comparatively small value of ζ_0, as already observed in the perturbation analysis.

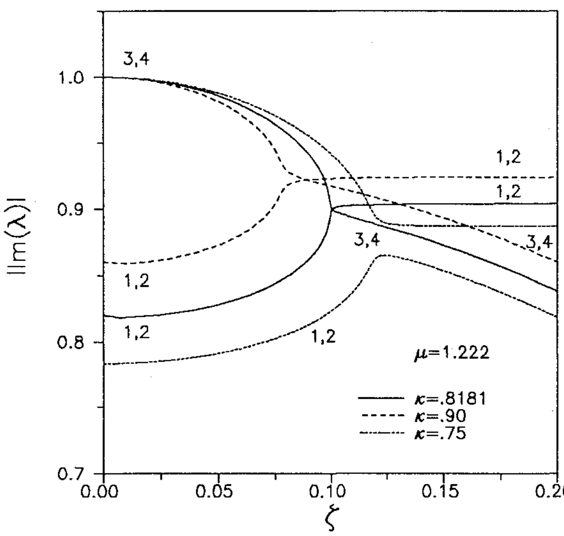
The associated eigenvectors $w(\zeta)$ are illustrated in Fig. 6 on the complex plane; components w_1 and w_2 are velocities, components w_3 and w_4 displacements, according to Eq. (4). When $\zeta = 0$, displacements and velocities of the two oscillators are uncoupled. When ζ increases, the components couple increasingly, but only the lengths of the vectors vary, whereas the phase differences remain practically unchanged. When $\zeta = \zeta_0 = 0.1$, length and phase differences in the two modes are identical, so that a unique eigenvector exists: the system has become defective. By further increasing ζ, the phase difference between w_3 and w_4 rapidly decreases in one mode and increases in the other one, whereas the lengths remain nearly constant. Therefore the two oscillators tend to vibrate in phase coincidence or in phase opposition, with small or large decay of the response, respectively. Velocities w_1 and w_2 exhibit similar behavior, the phase difference with respect to the associated displacement remaining greater than $\pi/2$. To conclude, eigenvectors are markedly sensitive to modifications, according to perturbation analysis.

It should be observed that the coalescence of the two eigenvectors for $\zeta = \zeta_0$ has been made possible by the fact that $\kappa = \kappa_0$ and $\mu = \mu_0$. If, for example, κ is slightly modified, μ being kept fixed, the loci $\lambda(\zeta; \kappa)$ shown in Fig. 5 by dashed lines are obtained. Now bifurcation does not occur [only $\mathrm{Re}(\lambda)$ or $\mathrm{Im}(\lambda)$ bifurcate], and the system remains non-defective for all values of ζ.

It is interesting to note a common aspect of the three systems examined: for given κ and μ an optimal damping ζ_{opt} exists for which the lowest exponential decrement $|\alpha| = |\mathrm{Re}(\lambda)|$ is a maximum. By increasing ζ over ζ_{opt}, the amplitude decaying becomes paradoxically slower. Moreover, when ζ is small, the smallest $|\alpha|$ corresponds to the lowest frequency $\omega = |\mathrm{Im}(\lambda)|$, whereas the opposite occurs when ζ exceeds (about) ζ_{opt}.



a)



b)

Fig. 5 Eigenvalues vs damping modifications.

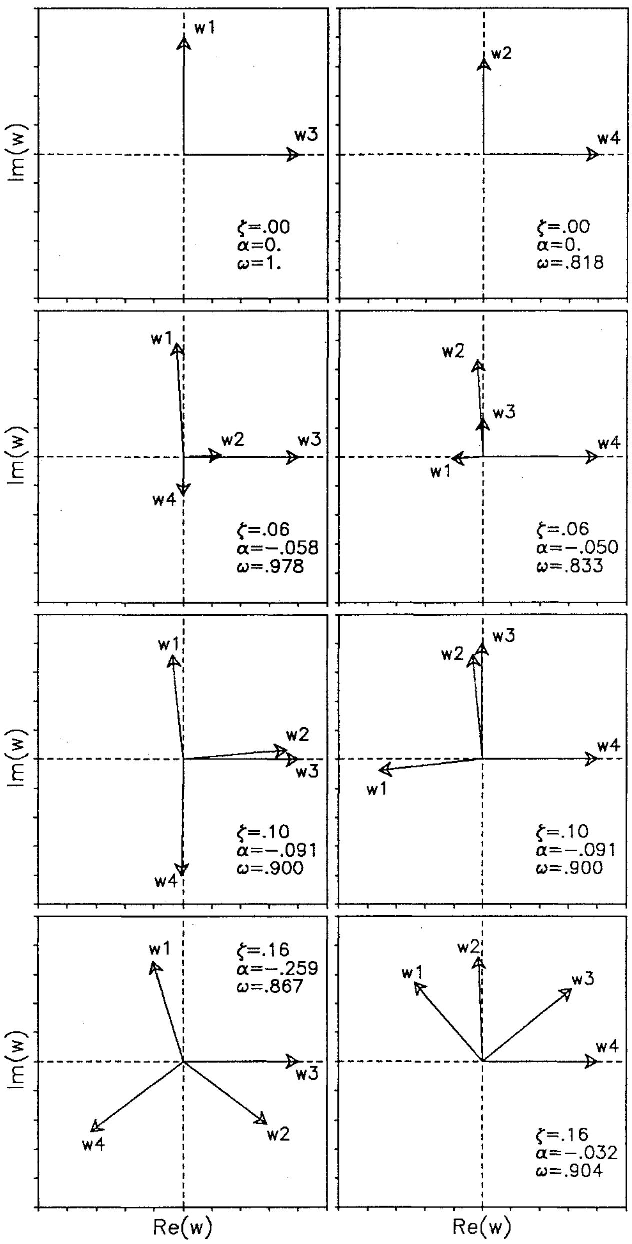


Fig. 6 Eigenvectors vs damping modifications.

B. Mass, Stiffness, and Damping Modification

All three system parameters are perturbed according to Eqs. (15). If it is wished to maximize the sensitivity (at least for small ζ_0), λ_1 is first calculated by Eq. (22), accounting for Eqs. (13) and (14) and for the expression (not reported here) of matrix A_1. It is found that

$$\lambda_1 = \left[\zeta_0(\kappa_1 - \mu_1 + 4\zeta_1)/2 + \mathcal{O}\left(\zeta_0^2\right)\right]^{\frac{1}{2}} \tag{29}$$

By choosing $k_1 = k_0$, $\mu_1 = -\mu_0$, and $\zeta_1 = \zeta_0$, then $\lambda_1 = \sqrt{\zeta_0} + \mathcal{O}(\zeta_0)$. In addition, Eqs. (15) read

$$\mu = \mu_0(1-\varepsilon), \qquad \kappa = \kappa_0(1+\varepsilon), \qquad \zeta = \zeta_0(1+\varepsilon) \tag{30}$$

It should be noted that in the previous subsection $\mu_1 = \kappa_1 = 0$ and $\zeta_1 = \mathcal{O}(\zeta_0)$, and so it ensues that $\lambda_1 = \mathcal{O}(\zeta_0)$. Therefore a higher sensitivity corresponds to modifications (30).

Figures 7–10 show the eigenvalues loci $\lambda(\varepsilon)$ for different values of ζ_0, obtained by varying ε in the interval $[-0.1, 0.1]$. Solid curves represent exact (numerical) solutions of problems (16), dashed curves perturbation solutions (19a) or (26a), at the first or second order of $\varepsilon^{1/m}$. Perturbative solutions have been obtained by numerically evaluating the generalized eigenvectors, also for small ζ_0; therefore, Eqs. (13) and (14) have not been used in the analysis.

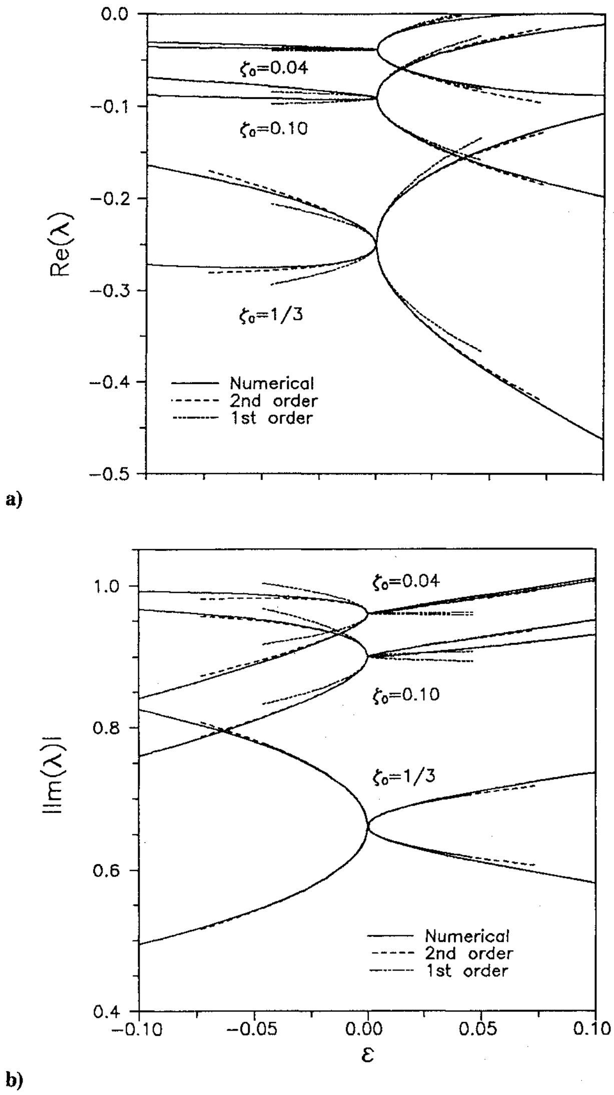
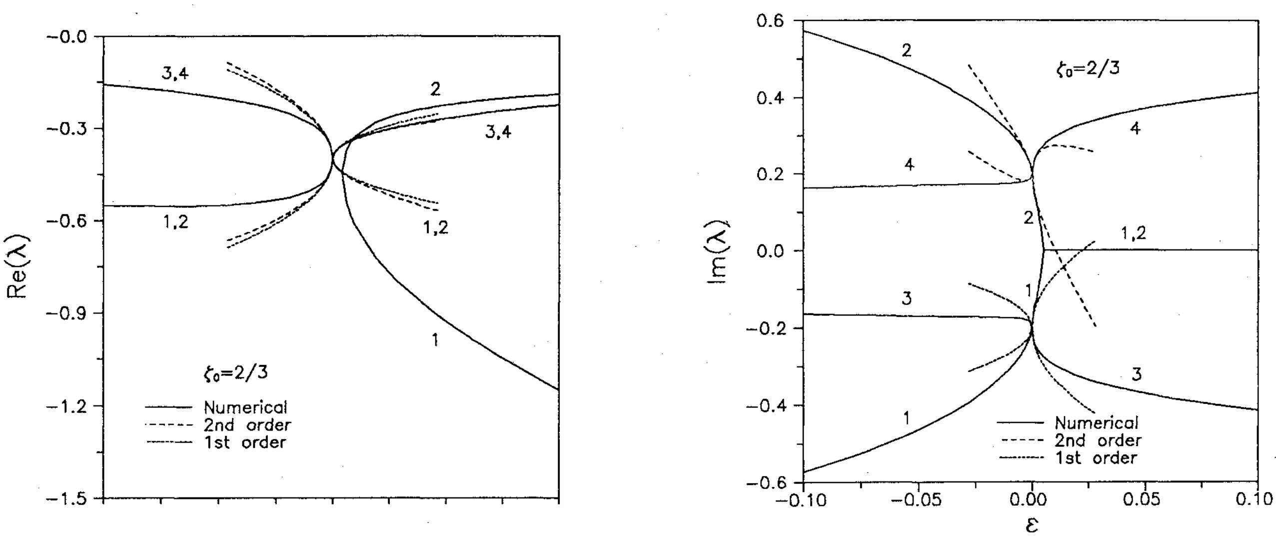


Fig. 7 Eigenvalues vs parameters modifications: underdamped systems.

Figure 7 concerns underdamped systems. It is seen that in the whole interval considered eigenvalues are complex conjugate. The curves exhibit singular points at $\varepsilon = 0$, which are bifurcation and limit points with respect to ε, simultaneously. Sensitivity is already noticeable when $\zeta_0 = 0.04$ and rapidly increases with ζ_0, according to the estimate $\lambda_1 \cong \sqrt{\zeta_0}$. The second-order perturbation approximation is excellent in a large neighborhood or $\varepsilon = 0$.

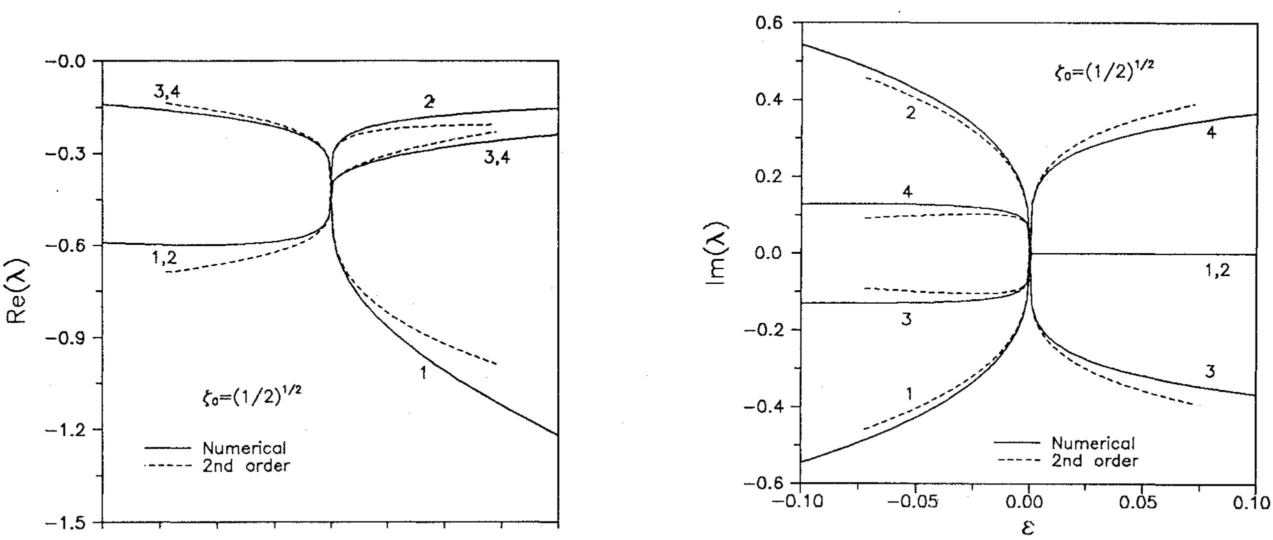
When the damping increases (Fig. 8), still remaining below the critical value $\zeta_{0_{cr}}$, sensitivity is magnified, but a new phenomenon becomes manifest. Indeed, the locus presents a secondary bifurcation point for $\varepsilon \cong 0.005$, where a couple of complex conjugate eigenvalues change into two separate real eigenvalues; thus, the modified system belongs to a new family of defective systems. This behavior obviously cannot be described by regular curves extrapolated from the primary bifurcation point (i.e., from the unmodified system); accordingly, the range of validity of the perturbation solution is limited. However, since the extrapolated eigenvalues loci intersect each other near the primary bifurcation point, the possible occurrence of a secondary bifurcation point is revealed by the perturbation method. Every time this circumstance occurs, the extrapolation should be repeated starting from a new regular point of the locus.

When the damping reaches its critical value (Fig. 9), the two bifurcation points, primary and secondary, coalesce. Sensitivity is much higher and is described excellently by the perturbation solution. It should be observed that however small the perturbations are, they modify one or two couples of real eigenvalues in complex conjugate, depending on the sign of ε.



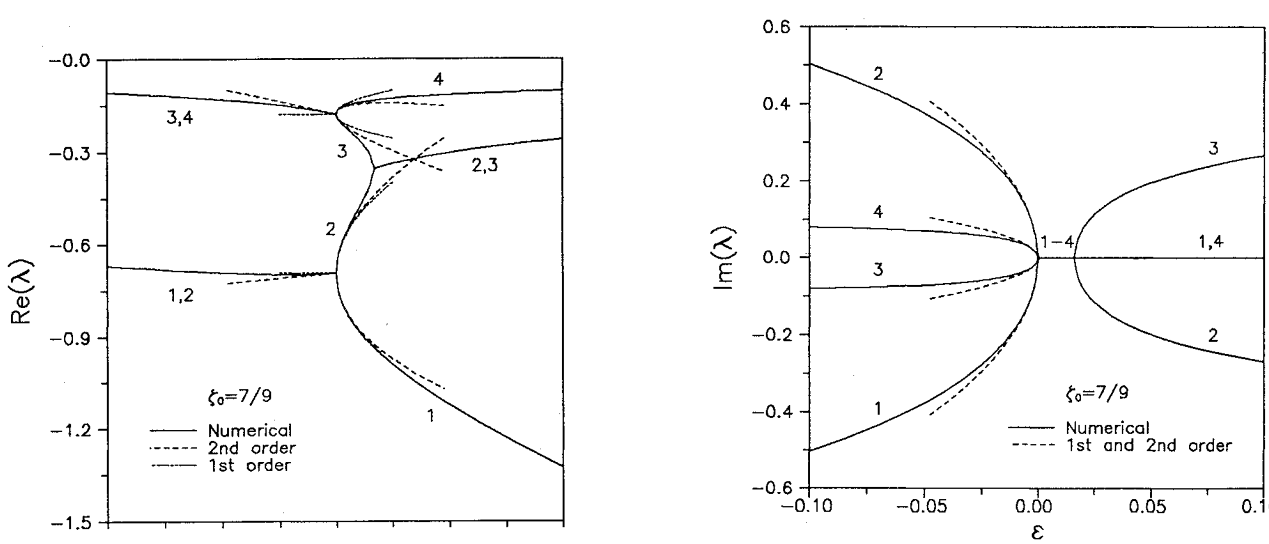
a) b)

Fig. 8 Eigenvalues vs parameters modifications: underdamped postbifurcating system.



a) b)

Fig. 9 Eigenvalues vs parameters modifications: critically damped system.



a) b)

Fig. 10 Eigenvalues vs parameters modifications: overdamped system.

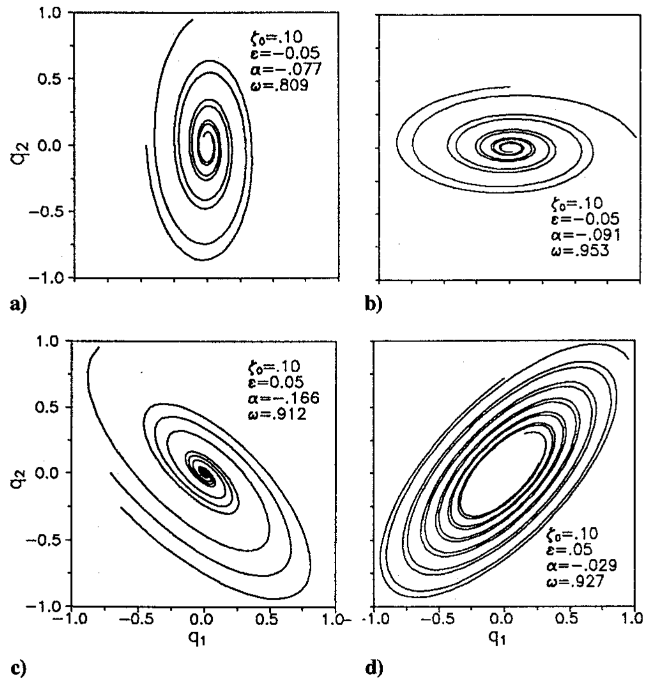


Fig. 11 Amplitude-decaying periodic motions of an underdamped oscillator.

In the case of the overdamped system (Fig. 10) it is seen that when $\varepsilon < 0$, however small it is, real eigenvalues become complex conjugate, whereas when $\varepsilon > 0$, they all remain real only if ε is less than (about) 0.016; indeed a secondary bifurcation occurs at this value of ε, giving rise to a couple of complex roots. Considerations similar to those made about Fig. 8 hold good.

An analysis of the effects of the modifications on the trajectories is presented in Fig. 11. Starting from the defective system $\zeta_0 = 0.1$, parameters have been modified according to Eqs. (30) by assuming $\varepsilon = \pm 0.05$. For the two modified systems some (numerical) trajectories for both modes have been plotted on the configuration variables plane [see Eq. (17)]. Curves should be compared with those in Fig. 2, relative to the unmodified system. In all cases the trajectories are strongly eccentric elliptical spirals, whose axes are parallel to the q_1 and q_2 axis when $\varepsilon < 0$ or form angles of $\pm\pi/4$ rad with them when $\varepsilon > 0$. Consequently, when $\varepsilon < 0$, one component prevails in each mode (Figs. 11a and 11b), and when $\varepsilon > 0$, the two components are roughly either in phase opposition (Fig. 11c) or in phase coincidence (Fig. 11d); in the latter case, motion decays very slowly. The analysis highlights marked sensitivity of the trajectories.

VI. Conclusions

Free oscillations and modal sensitivities of a one-parameter family of defective two DOF systems have been analyzed. In the four-dimensional state space the systems have only two complex (real) eigenvectors if slightly (heavily) damped or a unique real eigenvector if critically damped; therefore the base needs to be completed by generalized eigenvectors. The response is expressed as a sum of complex exponential motions and mixed exponential-algebraic motions. The trajectories are weakly attracted by the proper eigenvectors. If the damping ζ_0 is undercritical, the distance between trajectories and proper eigenvectors comes to be of order ζ_0 when the time is $\mathcal{O}(\zeta_0^{-2})$.

Small perturbations of order ε of the project parameters have next been introduced. By applying a perturbation method, the eigensolutions of the nearly defective system have been determined. Modal sensitivities of $\varepsilon^{\frac{1}{2}}$ order in the underdamped or overdamped cases have been found, whereas $\varepsilon^{\frac{1}{4}}$ order sensitivity appears in the critical case. The following qualitative conclusions have been drawn:

1) Lightly damped systems have moderately sensitive eigenvalues and strongly sensitive eigenvectors.

2) Heavily damped systems have strongly sensitive eigenvalues and moderately sensitive eigenvectors.

For given mass and stiffness ratios, an optimal value of the damping has been found for which maximum amplitude decaying occurs.

The parametric analysis performed has shown that small perturbations can give rise to real eigenvalues in the undercritical case and complex eigenvalues in the overcritical case. Consequently, the trajectories can be considerably modified by small perturbations. Few examples have been given.

The analysis also shows that defective systems can be very dangerous, since uncertainties or unavoidable imperfections can result in theoretical and real behavior being very different.

Acknowledgments

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References

[1]Brandon, J. A., *Strategies for Structural Dynamic Modification*, Wiley, New York, 1990.

[2]Luongo, A., "Eigensolutions Sensitivity for Nonsymmetric Matrices with Repeated Eigenvalues," *AIAA Journal*, Vol. 31, No. 7, 1993, pp. 1321–1328.

[3]Friedman, B., *Principles and Techniques of Applied Mathematics*, Wiley, New York, 1956.

[4]Wilkinson, J. H., *The Algebraic Eigenvalue Problem*, Clarendon Press, Oxford, England, UK, 1965.

[5]Kato, T., *A Short Introduction to Perturbation Theory for Linear Operators*, Springer-Verlag, New York, 1982.

[6]Pomazal, R. J., and Snyder, V. W., "Local Modifications of Damped Linear Systems," *AIAA Journal*, Vol. 9, No. 11, 1971, pp. 2216–2221.